Role of divergence of classical trajectories in quantum chaos

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We study logarithmic-in-$\hbar$ effects in the statistical description of quantum chaos. We found analytical expressions for the deviations from the universality in the weak localization correction and in the level statistics and showed that the characteristic scale for these deviations is the Ehrenfest time $t_E = \lambda^{-1}|\ln\hbar|$, where λ is the Lyapunov exponent of the classical motion. [S1063-651X(97)51101-9]

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It is accepted in the literature to call the consideration of quantum phenomena in classically chaotic systems ''quantum chaos'' [1]. For a de Broglie wavelength λ_F much smaller than the characteristic size of the system, quantum phenomena still bear essential features of the classical chaotic motion. Examples of such systems studied both theoretically and experimentally are ballistic cavities or antidot arrays [2]. The quantities usually considered include different correlators of quantum spectra of the system (level statistics) as well as of different response functions, e.g., fluctuations of the conductance (mesoscopics) or the quantum correction to the averaged transport coefficients (weak localization).

In principle, all the aforementioned characteristics can be found by solving the one-particle Schrödinger equation for the given system. However, the Schrödinger equation for such systems cannot be solved analytically. Substantial progress can be achieved in the statistical approach to quantum chaos. In such an approach one gives up attempts to find a contribution of a single quantum state but instead studies correlators averaged over large number of quantum states. The averaging for a given system can be performed either over wide range of energies or over the applied magnetic field.

In the present Rapid Communication we apply the supersymmetry description [3–5] to investigate how the universality is established in the statistical properties of the system at low frequencies. We will show that the time it takes to establish the universality is $t_E = \lambda^{-1}|\ln\hbar|$, where λ is the Lyapunov exponent of the classical motion. We will express deviations from the universality in the level statistics and in the weak localization corrections in terms of the single renormalization function [6]. Finally, we emphasize the necessity of the finite regulator in the Perron-Frobenius operator in obtaining physical results for the quantum corrections.

Let us first discuss the physical origin of the logarithmic-in-$\hbar$ corrections. In the semiclassical approximation, each classical trajectory corresponds to the quantum mechanical amplitude. Quantum phenomena in the system originate from the interference of the different amplitudes. After the averaging, most of the interference contributions vanish. Those that survive are the products that contain the pairs of coherent amplitudes. Such coherent amplitudes are contributed by the segments of the same classical trajectories and the resulting products are expressed in terms of the probabilities of finding such segments. The latter probabilities are found by solving the classical equation of motion. Most usable quantities are the probability where the initial i and final f states coincide $\mathbf{n}_f = \mathbf{n}_i, \mathbf{r}_f = \mathbf{r}_i$ [we will denote this probability as $\mathcal{D}_+(t;\mathbf{n}_i,\mathbf{r}_i)$] or are related to each other by time inversion $\mathbf{n}_f = -\mathbf{n}_i, \mathbf{r}_f = \mathbf{r}_i$ [we will denote this probability as $\mathcal{D}_-(t;\mathbf{n}_i,\mathbf{r}_i)$]. Here $\mathbf{r}$ and $\mathbf{n}$ are the coordinate of the particle and the direction of its momentum respectively. The first quantity is relevant for the leading approximation for the two point correlator of the density of states (DOS) [7,8,5,9], whereas the second quantity is important for the weak localization correction to the conductivity [6] and for the higher order approximations for the correlator of DOS; see below.

In what follows we will consider only ergodic systems. This means that after some time the particle visits all the phase space allowed by the energy conservation, i.e., the classical probabilities $\mathcal{D}_\pm$ averaged over the conditions $\mathbf{n}_i, \mathbf{r}_i$ cease to depend on time and take the value of $1/S$, where S is the volume of the system. It is very crucial that the equilibration time for $\mathcal{D}_-$ is parametrically larger than that for the probability $\mathcal{D}_+$.

The characteristic relaxation time for the probability $\mathcal{D}_+$ is of the order of the flying time of a particle across the system $\tau_{fl} \simeq L/v_F$ for the ballistic regime or the Thouless time $\tau_T \simeq L^2/D$ for the diffusive regime (L is size of the system, D is the diffusion coefficient, and v_F is the Fermi velocity).

On the other hand, in the strictly classical limit probability to have $\mathbf{n}_f = -\mathbf{n}_i, \mathbf{r}_f = \mathbf{r}_i$ vanishes no matter how large traveling time t is. This is due to the fact that the final state can be reached by moving along a classical trajectory which coincides with the initial one. This means that a particle must be reflected exactly backwards from an obstacle. For the chaotic system, the measure for such a process is zero and that is why $\mathcal{D}_-(t;\mathbf{n}_i,\mathbf{r}_i) = 0$. The only reason for this probability not to vanish is that the initial and final conditions cannot be specified with accuracy better than is allowed by the uncertainty principle. Due to this principle, the difference $|\mathbf{n}_f \times \mathbf{n}_i| = \delta\phi_0$ cannot be smaller than the diffraction spreading $\delta\phi_0 \gtrsim \sqrt{\lambda_F/a}$, with $a \gg \lambda_F$ being the characteristic spatial scale of the static potential the particle moves in. In order to

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find the probability for such close (but not coinciding) $\mathbf{r}_f,\mathbf{r}_i$, one has to take into account the fact that the motions of the particle at the initial and final stages are correlated. This is because the trajectory along which the particle moves on the final stage, $[\mathbf{r}(t-t_1),-\mathbf{n}(t-t_1)]$, almost coincides with the trajectory the particle moved along at the initial stage, $[\mathbf{r}(t_1),\mathbf{n}(t_1)]$. This problem is equivalent to the consideration of the divergence of two classical trajectories (''1'' and ''2'') which start from the same point $\mathbf{r}_i$ with a small difference in the directions of their momenta $|\mathbf{n}_2(0)\times\mathbf{n}_1(0)|=\delta\phi_0$ (it can be seen by the time inversion on the final segment). In the chaotic system the difference $\delta\phi_{12}(t)=|\mathbf{n}_2(t)\times\mathbf{n}_1(t)|$ grows exponentially with time $\delta\phi(t)\simeq\delta\phi_0 e^{\lambda t}$ where λ is the Lyapunov exponent of the classical chaotic motion. Therefore, we have also for the given trajectory $\delta\phi(t_1)=|\mathbf{n}(t-t_1)\times\mathbf{n}(t_1)|\simeq\delta\phi_0 e^{\lambda t_1}$. In order to close the trajectory at some time $t_1^*<t/2$, angle $\delta\phi(t_1^*)$ should become of the order of unity and thus $t\gtrsim(2/\lambda)\ln(1/\delta\phi_0)$. Taking into account $\delta\phi_0\gtrsim\sqrt{\lambda_F/a}$ we conclude that the time it takes to establish the equilibrium value of function $\mathcal{D}_-$ is the Ehrenfest time

$$t_E=\frac{1}{\lambda}\ln\left(\frac{a}{\lambda_F}\right), \tag{1}$$

and $\mathcal{D}_-$ vanishes at smaller time, $\mathcal{D}_-\simeq\theta(t-t_E)$.

The above discussion leads us to the following expressions for the Fourier transform of the classical probabilities $\mathcal{D}_\pm$ for the frequencies ω smaller than inverse time of the travel of the particle across the system:

$$\mathcal{D}_+(\omega)=\frac{1}{S}\frac{1}{-i\omega^+},\quad \mathcal{D}_-(\omega)=\frac{1}{S}\frac{\Gamma(\omega)}{-i\omega^+}, \tag{2}$$

where $\omega^+=\omega+i0$. The denominators in Eqs. (2) reflect the ergodicity of the system at large time and the renormalization function $\Gamma(\omega)$ describes the delay of $\mathcal{D}_-(t)$ with respect to $\mathcal{D}_+(t)$ by the Ehrenfest time t_E

$$\Gamma(\omega)=\exp\left(i\omega t_E-\frac{\omega^2\lambda_2 t_E}{\lambda^2}\right). \tag{3}$$

The second factor in Eq. (3) characterizes the fluctuations of the Lyapunov exponent and the parameter λ_2 is of the order of λ. More details about the derivation of function Γ can be found in Ref. [6]. Appearance of the new time scale t_E is the qualitative difference between quantum chaos $a\gg\lambda_F$ and quantum disorder $a\lesssim\lambda_F$ regimes. (In the systems where the scale of the potential a is different from the transport mean free path l_{tr}, the criteria for quantum chaos is $a\gg\sqrt{l_{tr}\lambda_F}$, see Ref. [6].)

A powerful method for the calculation of the averaged quantities is the supersymmetric nonlinear σ model pioneered by Efetov [3] for the disordered systems. Recently, the supersymmetric action was suggested by Muzykantskii and Khmelnitskii [4] and more recently by Andreev *et al.* [5] for the system in the ballistic regime. Effective action in Ref. [5] is defined by means of the classical Perron-Frobenius operator which differs from the first order Liouville operator by the regularizer of the second order. This approach enables one to perform the systematic semiclassical expansion for the averaged quantities and to understand how the underlying classical dynamics shows up in various quantum corrections.

Partition function $\mathcal{Z}$ in the supersymmetry approach is given by the functional integral [5]

$$\mathcal{Z}\{J\}=\int DQ(1)\exp\left[-\frac{\pi\nu}{2}\int d1\ \mathrm{STr}(\mathcal{L}+\mathcal{L}_J)\right], \tag{4}$$

with Lagrangians $\mathcal{L},\mathcal{L}_J$ being defined as

$$\mathcal{L}=\frac{i\omega^+}{2}\Lambda Q+T^{-1}\Lambda\hat{L}T+\frac{1}{4\tau}\left(\frac{\partial Q}{\partial\phi_1}\right)^2,\quad Q=T^{-1}\Lambda T$$

$$\mathcal{L}_J=iQ(1)\left[J_1 k\Lambda+(J_2(1)\Lambda_++J_2(\overline{1})\Lambda_-)\frac{k+1}{2}\right] \tag{5}$$

where ν is the density of states per unit area. We used the shorthand notation $1=(\mathbf{n}_1,\mathbf{r}_1),\overline{1}=(-\mathbf{n}_1,\mathbf{r}_1),d1=d\mathbf{n}_1 d\mathbf{r}_1/2\pi$, coordinate $\mathbf{r}$ and the direction of the momentum $\mathbf{n}=(\cos\phi,\sin\phi)$ characterize the position of the particle on the energy shell (we will restrict ourselves to two dimensional systems). Liouvillean operator $\hat{L}$ describes the classical evolution on the energy shell and defined by the Poisson bracket $\hat{L}\cdot=\{\cdot,\mathcal{H}\}$, where $\mathcal{H}$ is the Hamiltonian function. The last term in the Lagrangian $\mathcal{L}$ is the regularizer, the physical significance of which will be discussed later. The operation of supertrace is defined in Ref. [3]. We will consider only systems with the unbroken time reversal symmetry (orthogonal ensemble).

In Eq. (5), $\hat{T}$ is an 8×8 supermatrix defined in a linear superspace $p\otimes g\otimes d$ which we represent as the direct product of three linear spaces; p and d are the spaces of retarded-advanced and time reversal (complex conjugate) 2×2 matrices respectively, and g is the superspace of fermion-boson 2×2 supermatrices. All the relevant matrices can be conveniently expressed in terms of the Pauli matrices $\tau_z^\alpha,\tau_\pm^\alpha=(\tau_x^\alpha\pm i\tau_y^\alpha)/2$, acting in spaces $\alpha=p,g,d$. Matrices $\Lambda=\tau_z^p\otimes1^g\otimes1^d$, $\Lambda_\pm=\tau_\pm^p\otimes1^g\otimes1^d$, and $k=1^p\otimes\tau_z^g\otimes1^d$ break the symmetry in advanced-retarded and in fermion-boson spaces respectively. Matrix T is the subject to constraints $T^\dagger KT=K$ and $T^\dagger(1)=CT^T(\overline{1})C^T$, where $2K=[(1+\tau_z^p)\otimes1^g+(1-\tau_z^p)\otimes\tau_z^g]\otimes1^d$ and the matrix of charge conjugate C is given by $C=1^p\otimes(1^g\otimes\tau_-^d-\tau_z^g\otimes\tau_+^d)$.

Partition function (4) allows one to find different quantum mechanical correlators averaged over a wide range of energies. The two point correlator of the DOS, $R(\omega)=\Delta^2\langle\rho(\epsilon+\omega)\rho(\epsilon)\rangle_\epsilon$ [where $\rho(\epsilon)=\mathrm{Tr}\ \delta(\epsilon-\hat{H})$, with $\hat{H}$ being the Hamiltonian of the system, and Δ is the mean level spacing] and two-particle Green function $\mathcal{D}$ can be found as certain derivatives of action

$$R(\omega)=-\frac{\Delta^2}{16\pi^2}\mathrm{Re}\frac{\partial^2 Z}{\partial^2 J_1}\bigg|_{J_{1,2}=0}, \tag{6a}$$

$$\mathcal{D}(\omega;1,2)=-\frac{2\pi}{\nu}\frac{\delta^2 Z}{\delta J_2(1)\delta J_2(\overline{2})}\bigg|_{J_{1,2}=0}. \tag{6b}$$

We will be interested in effects associated with the energy scale $\omega \simeq t_E^{-1} \gg \Delta$, and will not consider contributions oscillating on Δ. For analysis in this situation it suffices to consider only small fluctuations of Q around Λ. We use the standard parametrization

$$T=\frac{1+iP}{\sqrt{1+P^2}}, \quad P\Lambda=-\Lambda P, \quad KP^{\dagger}K=P,$$

$$P(1)=-\overline{P(1)}, \tag{7}$$

where the operation of time reversal is defined as

$$\overline{M}=KCM^TC^TK, \tag{8}$$

for an arbitrary supermatrix M.

Substituting formula Eq. (7) into Eq. (5) and keeping terms up to the fourth order in P we obtain $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_{int}$ where the quadratic part of the Lagrangian describing the classical motion is given by

$$\mathcal{L}_0=P(-i\omega+\hat{L}_R)P, \quad \hat{L}_R=\hat{L}-\frac{1}{\tau}\frac{\partial^2}{\partial\phi^2}. \tag{9}$$

Operator $\hat{L}_R$ is known as the Perron-Frobenius operator. The quartic part, responsible for lowest order quantum corrections, has the form

$$\mathcal{L}_{int}=-P^3(-i\omega+\hat{L}_R)P+\frac{1}{\tau}\left(P\frac{\partial P}{\partial\phi}\right)^2. \tag{10}$$

We expand the partition function up to the first order in $\mathcal{L}_{int}$ and up to the second order in the source Lagrangian

$$\mathcal{L}_J=iJ_1k(1-2P^2+2P^4)+2J_2(k+1)\Lambda_+(P-P^3),$$

where we used Eq. (7) and omitted traceless terms. For calculating the averages of the arising products of matrices P we use the Wick theorem with the contraction rules

$$\begin{aligned}2\pi\nu\,\overbrace{P(1)MP}(2)=&\,\mathcal{D}^0(\bar{1},\bar{2})\Lambda_\parallel^+\mathrm{STr}[M\Lambda_\parallel^-]+\mathcal{D}^0(1,2)\Lambda_\parallel^-\mathrm{STr}[M\Lambda_\parallel^+]+\mathcal{D}^0(1,\bar{2})\Lambda_\parallel^-\bar{M}\Lambda_\parallel^+\\&+\mathcal{D}^0(\bar{1},2)\Lambda_\parallel^+\bar{M}\Lambda_\parallel^-,\\2\pi\nu\,\mathrm{STr}[M\overbrace{P(1)]\mathrm{STr}[NP}(2)]=&\,\mathrm{STr}[(\mathcal{D}^0(\bar{1},\bar{2})M-\mathcal{D}^0(\bar{1},2)\bar{M})\Lambda_\parallel^-N\Lambda_\parallel^+\\&+(\mathcal{D}^0(1,2)M-\mathcal{D}^0(1,\bar{2})\bar{M})\Lambda_\parallel^+N\Lambda_\parallel^-],\end{aligned} \tag{11}$$

where matrices $\Lambda_\parallel^\pm=(1\pm\Lambda)/2$ break the symmetry in the retarded-advanced subspace, and the classical propagator $\mathcal{D}^0$ is the solution of the equation

$$(-i\omega^++\hat{L}_R)_1\mathcal{D}^0(1,2)=2\pi\delta_{12}. \tag{12}$$

Substituting the result of the averaging in Eq. (6b), we obtain $\mathcal{D}(1,2)=\mathcal{D}^0(1,2)+\delta\mathcal{D}(1,2)$, where the quantum correction to the classical propagator is given by

$$\begin{aligned}2\pi\nu\,\delta\mathcal{D}(1,2)=&\,\mathcal{D}^0(1,\overline{2)}\mathcal{D}^0(\overline{2,2})+\mathcal{D}^0(1,\overline{1)}\mathcal{D}^0(\overline{1,2})\\&+\int d3\,\mathcal{D}^0(1,3)\mathcal{D}^0(\overline{3,2})(2i\omega-\hat{L}_R)_3\mathcal{D}^0(3,\overline{3)}.\end{aligned} \tag{13}$$

Notice that Eq. (13) gives a correction only to the nonzero modes of the Perron-Frobenius operator, $\int d1\,\delta\mathcal{D}(1,2)=0$, which is a consequence of the charge conservation.

Analogously, we obtain with the help of Eq. (6a) the following expression for the two-point correlator of DOS, $R=R^0+\delta R$. Here

$$R^0(\omega)=1+\frac{\Delta^2}{\pi^2}\mathrm{Im}\int d1\frac{\partial\mathcal{D}^0(1,1)}{\partial\omega}, \tag{14a}$$

which is the well-known result for the disordered [8] and chaotic systems [5]. The quantum correction has the form

$$\begin{aligned}\delta R(\omega)=\frac{\Delta^2}{2\pi^3\nu}\mathrm{Re}\Bigg[&4\pi\nu\int d1\; d2\;\mathcal{D}^0(1,2)\delta\mathcal{D}^0(2,1)\\&-\int d3\frac{\partial\mathcal{D}^0(\overline{3,3})}{\partial\omega}(2i\omega-\hat{L}_R)_3\frac{\partial\mathcal{D}^0(3,\overline{3)}}{\partial\omega}\Bigg].\end{aligned} \tag{14b}$$

Operator $\hat{L}_R$ in Eqs. (13) and (14b) acts on both arguments 3,3. Deriving Eqs. (14), we used the identity $-i\partial_\omega\mathcal{D}^0(1,2)=\int d3\,\mathcal{D}^0(1,3)\mathcal{D}^0(3,2)$.

Equations (13) and (14b) describe the lowest quantum corrections expressed in terms of the solutions of the Liouville equation (with the regularizer added) for a given system, where no ensemble averaging is assumed. They are both determined by the classical propagators between the points related by the time inversion. As we already explained, the equilibration time for such probabilities is the parametrically large Ehrenfest time.

Let us first discuss the two-point DOS correlator R. The classical propagator $\mathcal{D}^0$ entering into Eq. (14a) is the probability of the return to the initial state and at energies ω much smaller than the Thouless energy it coincides with $\mathcal{D}_+$ from Eq. (2). The first term in Eq. (14b) corresponds to the weak localization corrections of nonzero modes of the Perron-Frobenius operator and it can be neglected at such low frequencies. Propagators entering into the second term are the classical probabilities with the initial and final states related by the time inversion and they coincide with $\mathcal{D}_-$ at low frequencies. Substituting Eqs. (2) into Eqs. (14), we obtain

$$R(\omega)=1-\frac{\Delta^2}{\pi^2\omega^2}+\frac{\Delta^3\omega}{\pi^3}\mathrm{Im}\left(\frac{\partial}{\partial\omega}\frac{\Gamma(\omega)}{\omega}\right)^2+\cdots, \quad (15)$$

where the renormalization function Γ is defined in Eq. (3). It is easy to see that result (15) does not contain terms linear in t_E. Actually, this can be proven for all orders of perturbation theory in Δ/ω by using the approach similar to Ref. [11].

We emphasize that all the deviations from the universality of the nonoscillating part of $R(\omega)$ were studied in the diagonal approximation [5,7,8], and, therefore, they are associated with the time scale of the classical dynamics τ_{fl}. On the contrary, Eq. (14b) and the third term in Eq. (15) take into account the nondiagonal contribution and, thus, contain additional quantum smallness. Even though this correction to the universal Dyson result [10] for the orthogonal ensemble is small, it oscillates with period t_E^{-1}, where $\Delta\ll t_E^{-1}\ll\tau_{fl}^{-1}$ and, therefore, can be distinguished.

The result for the quantum correction to the conductivity of infinite system $\delta\sigma$ obtained from Eq. (13)

$$\delta\sigma(\omega)=-\frac{e^2}{2\pi^2\hbar}\ln\left(\frac{1}{\omega\tau_{tr}}\right)\Gamma^2(\omega)$$

is renormalized in comparison with the quantum disorder regime [12] by Γ. Details of the derivation of this equation from Eq. (13) can be found in Ref. [6].

The Ehrenfest time (1) contains the de Broglie wavelength λ_F. This scale is already absent in the effective σ model (5) which is formulated on the Hilbert space of functions smooth on scale λ_F, and the lower cutoff of the logarithm is related to the regularizer $1/\tau$ in the Lagrangian (5). It follows from the solution of Eq. (12) [6] that $\mathcal{D}_-$ has the form (2) with $t_E=\lambda^{-1}\ln(\lambda\tau)$. Therefore, the regulator cannot be put to zero even in the end of the calculation and it should be assigned some physical value. The value of the regularizer can be derived for the case if, in addition to the semiclassical potential, there are also quantum impurities in the system which provide the small angle scattering. The diffraction on the semiclassical potential itself is described by an equation more complicated than (12) and the regulator was not found consistently within the σ-model approach. However, it is not really necessary because the dependence on the regularizer is only logarithmical. Using Eq. (1) obtained by different arguments, we conclude that the physical value of the regularizer is given by

$$\frac{1}{\tau}=\lambda\frac{\lambda_F}{a}. \quad (16)$$

There is a subtlety in Eq. (13) which deserves more discussion because it helps to understand the importance of the regulator in the supersymmetric Lagrangian (5). With the help of Eq. (12), one can rewrite Eq. (13) in the more compact form

$$\delta\mathcal{D}(1,2)=\int d3\frac{\mathcal{D}^0(3,\overline{3})}{\pi\nu\tau}\frac{\partial\mathcal{D}^0(1,3)}{\partial\phi_3}\frac{\partial\mathcal{D}^0(\overline{2,3})}{\partial\phi_3}.$$

Naively, $\delta\mathcal{D}\to 0$ for $\tau\to\infty$. However, this contribution is anomalous and caution should be exercised while taking the limits. Namely, $\mathcal{D}^0(1,\phi_3+\delta\phi)\mathcal{D}^0(2,\phi_3-\delta\phi)$ is a singular function on $\delta\phi$ and this singularity is cut off by the same regulator $1/\tau$. As a result, the derivatives over ϕ_s are proportional to $\sqrt{\tau}$ and the dependence on the regularizer remains only logarithmical; see Eq. (13).

In conclusion, we studied logarithmic-in-$\hbar$ effects in the statistical description of quantum chaos. We found analytical expressions for the deviations from the universality and showed that the characteristic scale for these deviations is the Ehrenfest time $t_E=\lambda^{-1}|\ln\hbar|$, where λ is the Lyapunov exponent of the classical motion. Finally, we discussed the role of anomalies in the supersymmetric σ model [5] of quantum chaos.

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[1] *Chaos in Quantum Physics*, edited by M.-J. Jianonni, A. Voros, and J. Zinn-Justin, Les Houches, Session LII, 1989 (North-Holland, Amsterdam, 1991).
[2] See, e.g., H. U. Baranger, Physica D **83**, 30 (1995) and references therein.
[3] K. B. Efetov, Adv. Phys. **32**, 53 (1983).
[4] B. A. Muzykantskii and D. E. Khmelnitskii, Pis'ma Zh. Eksp. Teor. Fiz. **62**, 68 (1995) [JETP Lett. **62**, 76 (1995)].
[5] A. V. Andreev *et al.*, Phys. Rev. Lett. **76**, 3497 (1996); cond/mat 9605024.
[6] I. L. Aleiner and A. I. Larkin, Phys. Rev. B **54**, 14 423 (1996).
[7] M. V. Berry, Proc. R. Soc. London, Ser. A **400**, 229 (1985).
[8] B. L. Altshuler and B. I. Shklovskii, Zh. Éksp. Teor. Fiz. **91**, 220 (1986) [Sov. Phys. JETP **64**, 127 (1986)].
[9] E. B. Bogomolny and J. P. Keating, Phys. Rev. Lett. **77**, 1472 (1996).
[10] M. L. Mehta, *Random Matrices* (Academic, New York, 1991).
[11] V. E. Kravtsov and A. D. Mirlin, Pis'ma Zh. Éksp. Teor. Fiz. **60**, 645 (1994) [JETP Lett. **60**, 656 (1994)].
[12] L. P. Gorkov, A. I. Larkin, and D. E. Khmelnitskii, Pis'ma Zh. Éksp. Teor. Fiz. **30**, 248 (1979) [JETP Lett. **30**, 248 (1979)].